Conjugate of a complex number.

The conjugate of the complex number $x+iy$ is defined as the complex number $x-iy$.

The conjugate of z is denoted by $\bar{z}$

To get a conjugate of z replace i by $-i$ in z.

product of $z\bar{z}$ is a real number.

Properties of complex conjugates.

i) $\overline{z_1+z_2} = \bar{z}_1+\bar{z}_2$

ii) $\overline{z_1-z_2} = \bar{z}_1-\bar{z}_2$

iii) $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$

iv) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$, $z_2 \neq 0$.

v) $Re(z) = \frac{z+\bar{z}}{2}$

vi) $Im(z) = \frac{z-\bar{z}}{2i}$

vii) $\overline{z^n} = (\bar{z})^n$, $n \in \mathbb{Z}$.

(viii) z is real $\Leftrightarrow z=\bar{z}$

(ix) z is purely imaginary $\Leftrightarrow z=-\bar{z}$

(x) $\bar{\bar{z}} = z$

Exercise 2.4

1. Write the following in the rectangular form:

i) $\overline{(5+9i)+(2-4i)} = \overline{(5+9i)} + \overline{(2-4i)}$

$(\because \overline{z_1+z_2} = \bar{z}_1 + \bar{z}_2)$

$= (5-9i)+(2+4i)$

$= (5+2)+i(-9+4)$

$= 7-5i$

(ii) $\dfrac{10-5i}{6+2i} = \dfrac{(10-5i)}{6+2i} \times \dfrac{(6-2i)}{6-2i}$

$= \dfrac{60-20i-30i-10}{36+4}$

$= \dfrac{50-50i}{40}$

$= \dfrac{50}{40} - i\dfrac{50}{40}$

$= \dfrac{5}{4} - i\dfrac{5}{4} = \dfrac{5}{4}(1-i)$

(iii) $\overline{3i} + \dfrac{1}{2-i} = -3i + \dfrac{1}{2-i} \times \dfrac{2+i}{2+i}$

$= -3i + \dfrac{(2+i)}{4+1}$

$= -3i + \dfrac{(2+i)}{5}$

$= \dfrac{-15i+2+i}{5} = \dfrac{2-14i}{5}$

$= \dfrac{2}{5} - i\dfrac{14}{5}$

2. If $z = x + iy$, find the following in rectangular form.

(i) $Re\left(\frac{1}{z}\right) = Re\left(\frac{1}{x+iy}\right)$

$$= Re\left(\frac{1}{x+iy} \times \frac{x-iy}{x-iy}\right)$$

$$= Re\left(\frac{x-iy}{x^2+y^2}\right)$$

$$= Re\left(\frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}\right)$$

$$= \frac{x}{x^2+y^2}.$$

(ii) $Re(i\bar{z}) = Re(i(x-iy))$

$$= Re(ix + y)$$

$$= y$$

(iii) $Im(3z + 4\bar{z} - 4i)$

Let $z = x + iy$.
then $\bar{z} = x - iy$

$Im(3z + 4\bar{z} - 4i) = Im(3x + 3iy + 4x - 4iy -$

$= Im(7x + i(-y-4))$

$= -y-4$

(3) If $z_1 = 2-i$ and $z_2 = -4+3i$, find the inverse of $z_1 z_2$ and $\frac{z_1}{z_2}$

$$z_1 z_2 = (2-i)(-4+3i)$$
$$= -8+6i+4i+3$$
$$= -5+10i$$

$$(z_1 z_2)^{-1} = (-5+10i)^{-1} = \frac{1}{-5+10i}$$
$$= \frac{1}{-5+10i} \times \frac{-5-10i}{-5-10i}$$
$$= \frac{-5-10i}{(-5)^2-(10i)^2}$$
$$= \frac{-5-10i}{25+100} = \frac{-5-10i}{125}$$
$$= \frac{-5(1+2i)}{125}$$
$$= \frac{-(1+2i)}{25} = \frac{-1}{25} - \frac{2}{25}i$$

$$\frac{z_1}{z_2} = \frac{2-i}{-4+3i} \times \frac{-4-3i}{-4-3i}$$
$$= \frac{-8-6i+4i-3}{16+9}$$
$$= \frac{-11-2i}{25}$$

$$\left(\frac{z_1}{z_2}\right)^{-1} = \left(\frac{-11-2i}{25}\right)^{-1} = \frac{25}{-11-2i} \times \frac{-11+2i}{-11+2i}$$

Scriby Date ___ Page ___

$$= \frac{-275+50i}{121+4} = \frac{-275+50i}{125}$$

$$= \frac{25(-11+2i)}{125}$$

$$= \frac{-11+2i}{5}$$

$$= \frac{-11}{5} + \frac{2}{5}i$$

$$= \frac{1}{5}(-11+2i)$$

---

4. The complex numbers u, v and ω are related by $\frac{1}{u} = \frac{1}{v} + \frac{1}{\omega}$.

If $v = 3-4i$, $\omega = 4+3i$ find u in rectangular form.

$$\frac{1}{v} = \frac{1}{3-4i} = \frac{1}{3-4i} \times \frac{3+4i}{3+4i} = \frac{3+4i}{9+16} = \frac{3+4i}{25}$$

$$\frac{1}{\omega} = \frac{1}{4+3i} = \frac{1}{4+3i} \times \frac{4-3i}{4-3i} = \frac{4-3i}{16+9} = \frac{4-3i}{25}$$

$$\frac{1}{v} + \frac{1}{\omega} = \frac{3+4i}{25} + \frac{4-3i}{25} = \frac{7+i}{25}$$

$$\frac{1}{u} = \frac{7+i}{25}$$

$$u = \frac{25}{7+i} = \frac{25}{7+i} \times \frac{7-i}{7-i} = \frac{175-25i}{49+1}$$

$$= \frac{25(7-i)}{50} = \frac{7-i}{2} = \frac{7}{2} - \frac{1}{2}i$$

$$= \frac{1}{2}(7-i)$$

5. Prove the following properties

(i) z is real if and only if $z = \bar{z}$

(ii) $Re(z) = \frac{z+\bar{z}}{2}$; $Im(z) = \frac{z-\bar{z}}{2}$

<u>Proof:</u>

(i) Let $z = x+iy$

consider $z = \bar{z}$

$\Leftrightarrow \quad x+iy = x-iy$

$\Leftrightarrow \quad iy = iy$

$\Leftrightarrow \quad -2iy = 0$

$\Leftrightarrow \quad y = 0$

$\Leftrightarrow \quad z$ is real.

(ii) Let $z = x+iy$

$\bar{z} = \overline{x+iy} = x-iy$

$z+\bar{z} = x+iy+x-iy$

$= 2x$

$\frac{z+\bar{z}}{2} = x$

$= Re(z)$.

$z-\bar{z} = x+iy-(x-iy)$

$= x+iy-x+iy$

$= 2iy$

$\frac{z-\bar{z}}{2i} = y = Im(z)$

6. Find the least value of the positive integer n for which $(\sqrt{3}+i)^n$ (i) real (ii) purely imaginary.

Soln:

$(\sqrt{3}+i)^n$

If $n=1$ it is a complex number

Put $n=2$ $(\sqrt{3}+i)^2 = (\sqrt{3})^2 + 2\sqrt{3}i + (i)^2$

$= 3 + 2\sqrt{3}i - 1$

$= 2 + 2\sqrt{3}i$ = a complex n

Put $n=3$ $(\sqrt{3}+i)^3 = (\sqrt{3})^3 + 3(\sqrt{3})^2 i + 3(\sqrt{3})i^2 + i^3$

using $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$.

$= 3\sqrt{3} + 9i - 3\sqrt{3} - i = 8i$ = purely imagina

Now $(\sqrt{3}+i)^6 = (\sqrt{3}+i)^3(\sqrt{3}+i)^3$

$= 8i \times 8i$

$= -64$ = a real no.

$n=3 \Rightarrow (\sqrt{3}+i)^n$ is purely imaginary

$n=6 \Rightarrow (\sqrt{3}+i)^n$ is real

where n is the least positive integer.

7. Show that (i) $(2+i\sqrt{3})^{10} - (2-i\sqrt{3})^{10}$ is purely imaginary.

(ii) $\left(\frac{19-7i}{9+i}\right)^{12} + \left(\frac{20-5i}{7-6i}\right)^{12}$ is real

Proof:

i) $z = (2+i\sqrt{3})^{10} - (2-i\sqrt{3})^{10}$

$\bar{z} = \overline{(2+i\sqrt{3})^{10} - (2-i\sqrt{3})^{10}}$

$= \overline{(2+i\sqrt{3})^{10}} - \overline{(2-i\sqrt{3})^{10}}$ $(\overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2)$

$= \left(\overline{2+i\sqrt{3}}\right)^{10} - \left(\overline{2-i\sqrt{3}}\right)^{10}$

$= (2-i\sqrt{3})^{10} - (2+i\sqrt{3})^{10}$

$= -\left[(2+i\sqrt{3})^{10} - (2-i\sqrt{3})^{10}\right]$

$= -z$

$\bar{z} = -z \Rightarrow z$ is purely imaginary

(ii) $\frac{19-7i}{9+i} = \frac{19-7i}{9+i} \times \frac{9-i}{9-i}$

$= \frac{171 - 19i - 63i - 7}{81+1}$

$= \frac{164 - 82i}{82} = 2 - i$

$\frac{20-5i}{7-6i} = \frac{20-5i}{7-6i} \times \frac{7+6i}{7+6i}$

$$= \frac{140 + 120i - 35i + 30}{49 + 36}$$

$$= \frac{170 + 85i}{85} = 2 + i$$

$$z = \left(\frac{19-7i}{9+i}\right)^{12} + \left(\frac{20-5i}{7-6i}\right)^{12} = (2-i)^{12} + (2+i)^{12}$$

Now $\bar{z} = \overline{(2-i)^{12} + (2+i)^{12}}$

$$= \overline{(2-i)^{12}} + \overline{(2+i)^{12}}$$

$$= (\overline{2-i})^{12} + (\overline{2+i})^{12}$$

$$= (2+i)^{12} + (2-i)^{12}$$

$$= z$$

$\bar{z} = z \Rightarrow z$ is real.

Hence Proved.